Turán's Theorem

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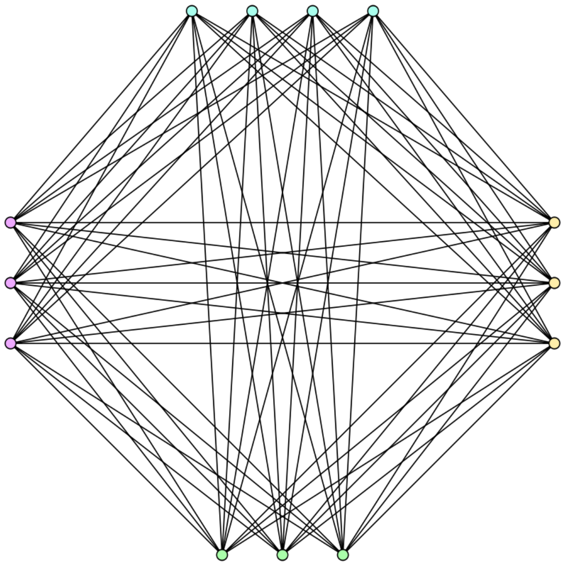

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Chapter 1

Introduction

Graph theory is the study of graphs, simply said, dots and the lines that connect them. Graphs are structures that model pairwise relations. Through graph theory, many practical problems can be both defined and represented, showcasing its usefulness through modeling relations in biological, physical, social, and information systems. Graph theory is one of the basic subjects of study in discrete mathematics, which started with a famous paper written by Leonhard Euler on the Seven Bridges of Königsberg problem. [1]

Extremal graph theory is a branch of mathematics that explores how certain graph properties like the number of vertices, edges, edge density, etc. impact properties of local subcultures. Mantel's theorem and Turán's theorem were one of the first defining discoveries that have shaped the course of the study of extremal graph theory. Erdös wrote of Turán, "In 1940–1941 he created the area of extremal problems in graph theory which is now one of the fastest-growing subjects in combinatorics." [2]

1.1 Fundamentals

- A graph is formed by vertices (alternatively called nodes, indices) and edges connecting them i.e. a graph is a pair of sets $G(V, E)$.
- The degree of a vertex v, written as $d(v)$, is the number of edges with v as an end vertex.
- Two vertices connected by an edge are called neighbors.
- A subgraph of graph G that is induced by all vertices connected to v is called the neighborhood of v. The neighborhood comprises all of v's neighbours and the edges joining them.
- Edges that share two same end vertices are called parallel, edges that begin and end in the same vertex are called loops.

- A graph not containing parallel edges and loops is called a simple graph. In this paper, we will strictly work with simple graphs. [3]
- The maximum number of edges that a graph G that has n vertices can have is $\binom{n}{2} = \frac{n(n-1)}{2}$. Such a graph is called a complete graph, one where every possible edge is drawn.
- A clique is a subset of a graph G, where every vertex is connected to every vertex from that subset, i.e the induced subgraph is a complete graph.
- An important theorem is that the sum of degrees of all nodes in a graph is twice the number of edges, $\sum_v deg(v) = 2|E|$. This observation is left as an exercise for the reader :). [4]

1.2 Notations

$V(G)$	the vertex set of graph G
$E(A, B)$	edges between sets A and B
$E(G)$	the edge set of graph G
$d(v)$	degree of vertex v
$\omega(G)$	largest clique of graph G
$\delta(G)$	largest degree in G
$N_G(v)$	neighbourhood of vertex v
α	independence number
$:=$	is defined as
$\|A\|$	size of set A
A/B	difference between set A and B
$\sum$	summation
$\lfloor k \rfloor$	floor function
$\lceil k \rceil$	ceiling function
$\binom{n}{k}$	n choose k
$\longleftrightarrow$	if and only if
$\therefore$	therefore
$WLOG$	without loss of generality
$\blacksquare$	quod erat demonstrandum

Chapter 2

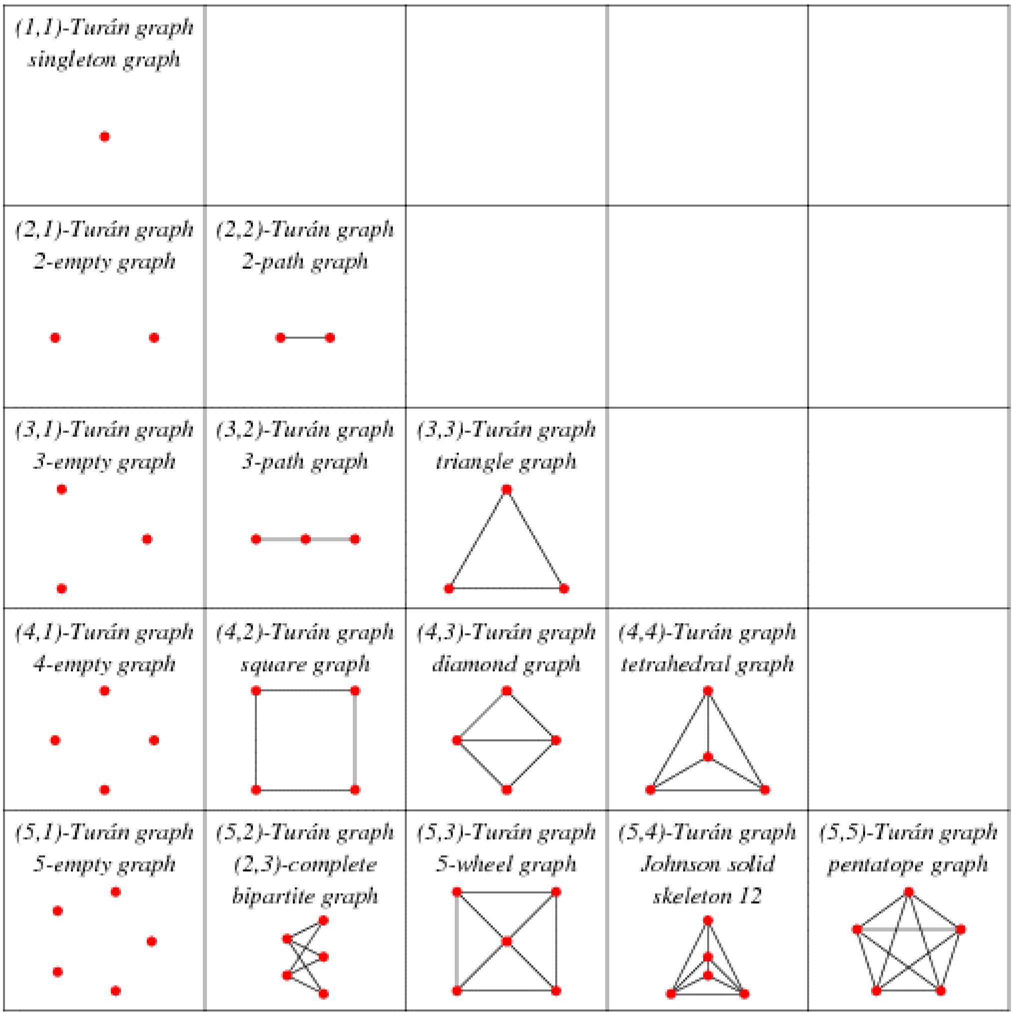
Turán Graph

Pál Turán asked the question: If G is a simple graph not containing a p-clique, how large can $|E(G)|$ be?

We can easily obtain such an example if we divide $V(G)$ into $p-1$ disjoint subsets V_1, V_2, $\ldots$, V_{p-1}, $|V_i| = n_i$ $(n = n_1 + n_2 + \ldots + n_{p-1})$ and we strictly join two vertices if they lie in distinct sets V_i, V_j. (Graph G is usually denoted by $K_{n_1,n_2,..,n_{p-1}}$.) It is clear that $|E(G)| = \sum_{i \leq j} n_i n_j$. We get the maximum $|E(G)|$ for given n, if we divide the nodes as evenly as possible (uniformly spread), hence $|n_i - n_j| \leq 1$ for all i,j. In fact, let's suppose that $n_1 \geq n_2 + 2$. With moving one node from V_1 to V_2 we obtain graph $G\prime = K_{n_1-1,n_2+1,..,n_{p-1}}$, which results in

$$|E\prime| - |E| = (n_1 - 1)(n_2 + 1) - n_1 n_2 = n_1 - n_2 - 1 \geq 1 > 0$$

Graphs of the form $K_{n_1,n_2,..,n_{p-1}}$ with $|n_i - n_j| \leq 1$ are called Turán graphs. If they are formed by partitioning a set of n vertices into r subsets, then they can be denoted by $T(n, r)$. [5]



Examples for small values of n

Chapter 3

Mantel's Theorem

Theorem (**Mantel, 1906**). *Let G be a triangle-free simple graph. If $n = |V(G)|$ and $m = |E(G)|$, then $m \leq \dfrac{n^2}{4}$. Equality holds if and only if $G = K_{\frac{n}{2},\frac{n}{2}}$ for an even n.* [6]

First Proof. Let x and y be two vertices in G joined by an edge. If $d(v)$ is the degree of vertex v, then $d(x) + d(y) \leq n$. This is because vertices x and y cannot share a neighbor, meaning that all vertices in G are at most connected to one of x and y.

$$\sum_x d^2(x) = \sum_{xy \in E} (d(x) + d(y)) \leq mn$$

Simultaneously, we have that $\sum_x d(x) = 2m$, and by the Cauchy-Schwarz inequality we have that:

$$\frac{4m^2}{n} \leq \frac{(\sum_x d(x))^2}{n} \leq \sum_x d^2(x)$$

$\therefore$

$$\frac{4m^2}{n} \leq mn \Rightarrow m \leq \frac{n^2}{4}$$

■

Second Proof. We proceed by induction on n. The cases for $n = 1$ and $n = 2$ are trivial. Let's assume that the statement is true for $n - 1$. Let x and y be two vertices in G joined by an edge. As above, we know that $d(x) + d(y) \leq n$. Let S be the subset of G that contains all vertices except x and y, i.e. $|S| = n - 2$. Since S contains no triangles it must, by induction, contain at most $\dfrac{(n-2)^2}{4}$ edges.

$\therefore$

$$m = E(S) + E(x) + E(y) - 1 \leq \frac{(n-2)^2}{4} + n - 1 = \frac{n^2}{4}$$

■

Third Proof. Let α be the independence number of G i.e. α is the size of the largest subset S from $V(G)$ so that $e(S) = 0$. We denote $\beta = n - \alpha$. For every $v \in V(G)$,

$$|N_G(v)| \leq \alpha$$

In other words, for every $v \in V(G)$ the degree $d(v) \leq \alpha$. Because S is an independent set in G, every edge has at least one end in set $T = V(G) \backslash S$. If we count $E(G)$ from T we get that

$$|E(G)| \leq \sum_{v \in T} d(v)$$

$\therefore$

$$|E(G)| \leq \alpha\beta \leq (\frac{\alpha + \beta}{2})^2 \leq \frac{n^2}{4}$$

Equality holds iff $\alpha = \beta = \frac{n}{2}$, every vertex from T has degree $\frac{n}{2}$ and T is independent (Equality holds iff $G = K_{\frac{n}{2},\frac{n}{2}}$). For n odd, $E(G)$ is maximised when $\alpha = \lceil \frac{n}{2} \rceil$ and $\beta = \lfloor \frac{n}{2} \rfloor$. ■

This proof shows that not only is $\lfloor \frac{n^2}{4} \rfloor$ the maximum number of edges in a triangle-free graph but also that any triangle-free graph with this number of edges is bipartite with partite sets of almost equal size.
The natural generalisation of this theorem to cliques of size p-1 is Turán's theorem. [7]

Chapter 4

Turán's Theorem

Theorem (**Turán, 1941**). *Let $G = (V, E)$ be a simple graph with n vertices so that $\omega(G) < p$, where $p \geq 2$. Then*

$$|E| \leq (1 - \frac{1}{p-1})\frac{n^2}{2}$$

Equality holds if and only if G is a Turán graph. [8]

The case p=2 is trivial. The first non-trivial case p=3 is Mantel's theorem which was proved prior to Turan's result. Returning to the general result, the first two proofs are by Pál Turán and Pál Erdös, respectively.

First Proof. We proceed by induction on n. It is evident that the theorem is true for $n < p$: due to $|E| \leq \binom{n}{2} = \dfrac{n(n-1)}{2}$, while $(1 - \dfrac{1}{p-1})\dfrac{n^2}{2} \geq (1 - \dfrac{1}{n})\dfrac{n^2}{2} = \dfrac{n(n-1)}{2}$. Let G be a simple graph on $V = \{v_1, ..., v_n\}$ so that $\omega(G) < p$, $n \geq p$ and G has the maximum number of edges. It is clear that $\omega(G) = p - 1$ (otherwise we would be able to add edges while still satisfying the inequality $\omega(G) < p$). Let A be a $(p-1)-$clique and set $B := V \backslash A$. It follows that

$$E(A) = \binom{p-1}{2}$$

and by the induction hypothesis

$$E(B) \leq \frac{(1 - \frac{1}{p-1})(n - (p-1))^2}{2}$$

Because there isn't a p-clique in G, every $v \in B$ has at most $p-2$ neighbours in A. Hence,

$$E(A, B) \leq (p-2)(n - (p-1))$$

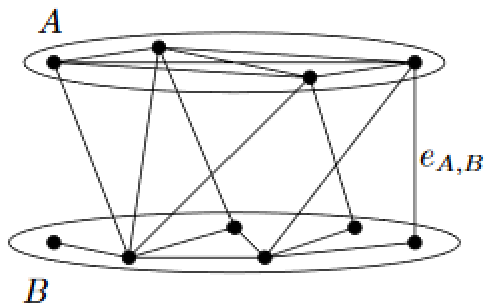
We get that

$$|E| = E(A) + E(B) + E(A, B) \leq$$

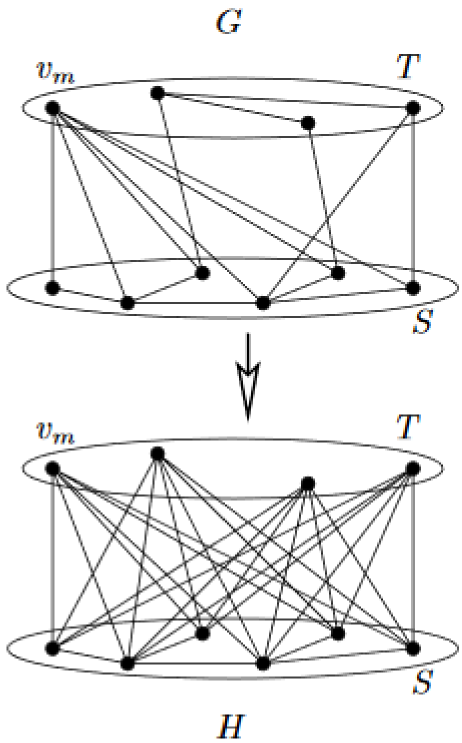
$$\leq \binom{p-1}{2} + \frac{(1-\frac{1}{p-1})(n-(p-1))^2}{2} + (p-2)(n-(p-1)) =$$

$$= \frac{(1-\frac{1}{p-1})(n^2)}{2}$$

■



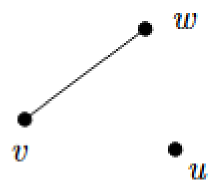
Second Proof. Let $v_m \in V = \{v_1, ..., v_n\}$ be a vertex with the greatest degree, we denote $d_i = d(v_i)$. In other words, $d_m = \delta(G) = max_{1\leq i\leq n} d_i$. Let $S = N_G(v_m)$ and set $T := V \backslash S$. Because G doesn't contain a p-clique, S doesn't contain a (p-1)-clique. We construct a new graph $G\prime$ on V (see figure) by the following: G and $G\prime$ correspond on S, i.e. $e(S) = e\prime(S)$; furthermore, $G\prime$ contains all possible edges between S and T, i.e. $e\prime(S,T) = |S||T|$; however, $G\prime$ doesn't contain any edges within T, $e\prime(T) = \varnothing$. T is an independent set in $G\prime$ and $\omega(G\prime) < p$. Let's denote $d_{G\prime}(v_j)$ with $d\prime_j$. We have the following options: if $v_j \in S$, then $d\prime_j \geq d_j$ (based on the construction of $G\prime$) and if $v_j \in T$, then $d\prime_j = |S| = d_m \leq d_j$ (based on the choice of v_m). We deduce that for every $v \in V$, $d\prime(v) \geq d(v)$ and consequently $|E\prime| \geq |E|$, finding that among simple graphs on V that have $\omega < p$ and maximize $|E|$, there must exists a graph of the form of $G\prime$. By induction on p, the graph induced by S cannot have more edges than at least one graph $Kn_1, n_2, ..., n_{p-1}$ on S. So $|E| \leq |E'| \leq E(Kn_1, n_2, ..., n_{p-1})$ with $n_{p-1} = |T|$, which implies the theorem. ■



The last proof is often regarded as the most beautiful proof of Turán's theorem and has unknown origins. It is dubbed as "folklore" graph theory.

Third Proof. Let G be a simple graph on n vertices so that $\omega(G) < p$ and $|E|$ is maximized under the given conditions.

Claim: G does not contain three vertices u, v, $w \in V(G)$, such that $vw \in E(G)$, but $uv \notin E$, $uw \notin E$.

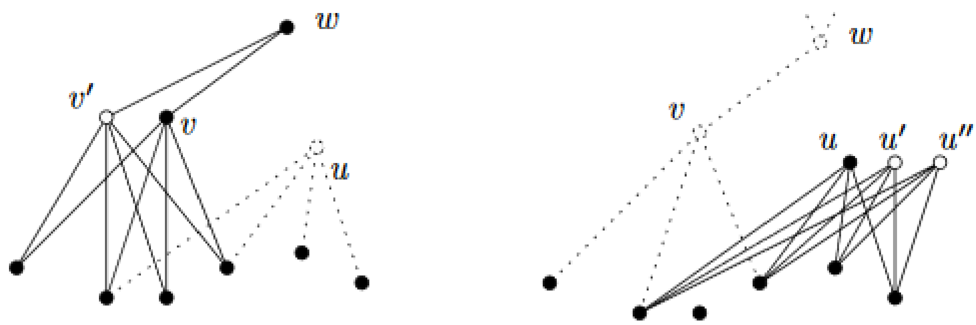


Proof: Let's suppose the contrary. We observe two cases.
(i) $d(u) < d(v)$ or $d(u) < d(w)$
WLOG, let $d(u) < d(v)$, we "duplicate" v, we create a copy $v\prime$ which will have the same

neighbors as v, while $vv\prime \notin E(G)$, we "delete" u and everything else remains untouched. The newfound graph $G\prime$ also doesn't contain a p-clique and we find that:

$$|E\prime| = |E| + d(v) - d(u) > |E|$$

a contradiction!



(ii) $d(u) \geq d(v)$ and $d(u) \geq d(w)$
For this case, we "triple" u and we "delete" v and w. In other words, we add two copies, $u\prime$ and $u\prime\prime$, who have the same neighbours as u, while $uu\prime$, $uu\prime\prime$, $u\prime u\prime\prime \notin E(G)$. The newfound graph $G\prime$ also doesn't contain a p-clique and we find that:

$$|E\prime| = |E| + 2d(u) - (d(v) + d(w) - 1) > |E|$$

a contradiction, again. This proves our claim. ■

An equivalent form of the previous claim is the following claim about a binary relation $\sim$ on G, defined as:

$$u \sim v \longleftrightarrow uv \notin E(G)$$

Claim: $\sim$ is an equivalence relation.

Proof: An equivalence relation is reflexive, symmetric and transitive. G does not contain loops, hence $\sim$ is reflexive. By definition, $uv \in E(G)$ iff $vu \in E(G)$, hence $\sim$ is symmetric. By the previous claim we get that $\sim$ is transitive. Satisfying all three properties proves the claim. ■

Therefore, $\sim$ provides a partition of $V(G)$ into disjoint equivalence classes. This means G has to be a complete multipartite graph, $Kn_1, n_2, ..., n_{p-1}$. ■

All given proofs also yield a stronger result: that the Turán graph is the unique example with the maximal number of edges, satisfying the equality of the theorem.

References

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[5]A. Shapira, *Extremal graph theory* (2016), p. 4.

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[7]S. Kopparty, *Local structure: forbidden subgraphs* (2011), p. 1.

[8]G. M. Z. Martin Aigner, *Proofs from the book* (2013), pp. 235, 236, 238.